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A BIFURCATION MODEL OF THE QUANTUM FIELD

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The paper reveals a new interpretation of the Standard Model for elementary particle physics. The approach is based on the concept of chaotic behavior applied to the gauge transformation. Following the framework of bifurcation theory, the paper provides a simple and consistent picture of lepton and boson production.

The author reveals the fractal structure of the quantum harmonic oscillator. A map of quantum field organisation is developed together with the Lagrangian formalism of the model. On this map the structure of the field appears as a "branching out" pattern.

The paper describes the internal symmetry between gravity and electromagnetism with respect to the gauge transformation. A discussion on the physical meaning of the W field and of the isospin eigenstate T_3 is included.

1. Introduction

A major achievement in the field of elementary particle physics has been the development of the Standard Model (SM). As is currently accepted, this model represents a unique synthesis of the theory of the strong force (QCD) with the theory of electroweak interaction. QCD is basically a non-Abelian gauge theory on the SU(3) color while the electroweak model successfully describes the interaction of quarks and leptons.

Although a formally consistent approach, SM is far from being completed. There are several open questions that await clarification and further extensions are yet to come. Among the issues that are not covered by the SM, the key ones are as follows:

(A) Why are there three families of gauge bosons?
(B) Why are there three flavors of neutrinos and why are they left-handed?
(C) Why six quarks and eight gluons?
(D) Is there a physical background underlying the spontaneous breaking of symmetry and is the Higgs mechanism the only way to understand the mass spectrum?

The present paper is intended to fill in some of the remaining "holes" in the conceptual structure of the SM.

The approach is based upon the geometry of fractals and the theory of bifurcations applied to nonlinear dynamical systems. As stated, our goal is to deepen the understanding of SM without altering its fundamental construction. Consequently, our theory suggests an alternative route to SM.

Fractals have found wide application in many forefront areas of physics such as condensed matter, crystals growth, polymer statistics, fracture propagation, physics of spin systems and so on.

An analysis of the quantum harmonic oscillator performed in phase space reveals that it is possible to assign a fractal configuration to this space. Since the scale invariant properties of fractals are intimately connected with the chaotic behavior of nonlinear systems, it makes sense to further investigate how these concepts can be incorporated in the SM.

The physical foundation of this treatment lies in the following:

(A) All elementary particles (leptons, quarks and gauge bosons)–taken as solutions of the quantum field equations–are essentially nonlinear dynamical systems.

(B) The bifurcation theory–as developed by Feigenbaum in 1977–claims universality over the internal evolution of nonlinear dynamical systems.

As a result, there is a specific instability associated with the equations describing the field transformations. It is shown that this mechanism can unveil the spectrum of field quanta in a sequential manner.

The paper develops by applying the theory of bifurcations directly to the gauge transformation. The gauge transformation is the backbone of the whole SM construction. It is the nonabelian gauge invariance that makes the SM a renormalizable theory and determines the phenomenological structure of it [1].

The main benefit of this approach lies in the fact that particles appear organized in a regular and self-similar pattern. The quantum field has a structure which repeats itself from family to family. This "branching-out" layout is consistent with the nonabelian symmetries of the SM and brings into a unique picture all known bosons and fermions.

A possible extension of the bifurcation model is also outlined. This formalism explores the derivation of vacuum expectation values for the field in a manner that bypasses the Higgs mechanism. As mentioned earlier, the framework of the Higgs mechanism is not presently understood.

The paper finally examines a possible extension of the bifurcation model to include the gravitational field in the picture. The underlying physics relates to the internal symmetry of gravity and electromagnetism with respect to the gauge transformation. A discussion on the physical meaning of the W boson and of the isospin T_3 is presented. Several postulates are introduced to make

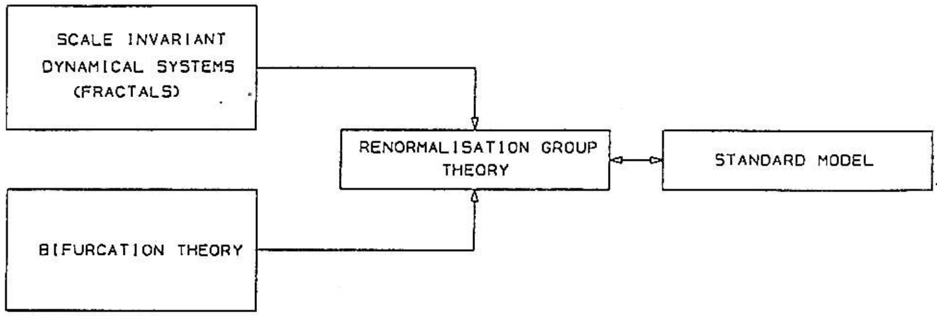

Fig. 1.

the treatment self-contained and to set its limits of validity. The rationale of each postulate is briefly reviewed below:

(P1). This postulate originates in the uncertainty principle and sets the quantum-mechanical zero point fluctuations of canonical variables. Therefore it has to be understood as a definition of the first order differential from the standpoint of the measurement process.

(P2). This postulate states the definition of a "noise-free" representation of the theory and originates also in the measurement process of canonical variables.

(P3). This postulate is a consequence of the "charge" conservation theorems from the relativistic quantum field theory.

(P4), (P5). These are transcriptions of the exclusion principle and of the Schwinger–Lüders–Pauli theorem, respectively.

Our approach is part of a general effort of theorists to broaden the knowledge basis of SM. Its relationship with the SM lies mainly in the so-called Renormalization Group ideas [1, 2] as implied by the diagram in fig. 1.

2. A fractal treatment of the harmonic oscillator

This section is an attempt to prove that the one-dimensional vibrational mode can be represented as a fractal object. As is well known, the linear harmonic oscillator is described by canonical operators p and q satisfying the commutation relation

$$[q, p] = \mathrm{i} . \tag{1}$$

The analogous classical system exhibits elliptical orbits in the phase space,

$$\frac{p^2}{2mE} + \frac{q^2}{(2E/m\omega^2)} = 1\,, \tag{2}$$

where E is the total energy and m is the reduced mass of the particle. The area determined by the orbit boundary is always a finite number:

$$S = \int_D p\,\mathrm{d}q = n + \tfrac{1}{2}\,, \tag{3}$$

n being a finite positive integer (Sommerfeld's quantisation rule). To evaluate the length of the orbit boundary we introduce two postulates as follows.

⟨P1⟩. The first order differential of canonical variables has the same order of magnitude as their respective fluctuations [3],

$$\mathrm{d}q = \Delta q = (2\omega)^{-1/2}\,, \tag{4}$$

$$\mathrm{d}p = \Delta p \geqslant (2\,\Delta q)^{-1}\,. \tag{5}$$

⟨P2⟩. The coordinate fluctuation is smaller than the coordinate itself:

$$\Delta q/q = 1/q(2\omega)^{1/2} < 1\,. \tag{6}$$

The above statement takes into account the fact that measurement of q is supposed to provide an acceptable level of resolution. Only this circumstance is able to generate a "noise-free" representation of the field behaviour. The length of the orbit is the line integral given by

$$L = 4 \int_0^{(2E/m\omega^2)^{1/2}} \left[1 + \left(\frac{\mathrm{d}p}{\mathrm{d}q}\right)^2\right]^{1/2} \mathrm{d}q\,. \tag{7}$$

Let us replace now the orbit slope by the ratio of coordinate and momentum fluctuations and use postulate ⟨P2⟩,

$$L \geqslant 4 \int_0^{(2E/m\omega^2)^{1/2}} [1 + \tfrac{1}{4}(\Delta q)^{-4}]^{1/2}\,\mathrm{d}q \geqslant 4 \int_0^{(2E/m\omega^2)^{1/2}} [1 + \tfrac{1}{4}(q)^{-4}]^{1/2}\,\mathrm{d}q \tag{8}$$

$$\to \frac{L}{2} \geqslant \int_0^{(2E/m\omega^2)^{1/2}} q^{-2}(1 + 4q^4)^{1/2}\,\mathrm{d}q = \int_0^1 \cdots + \int_1^{(2E/m\omega^2)^{1/2}} \cdots\,. \tag{9}$$

This definite integral can be computed starting from the general form [4]:

$$J=\int_0^1 x^{p-1}(1-x)^{n-1}(1+bx^m)^l \, dx \qquad (b^2>1)\,, \tag{10}$$

where

$$p=-1\,, \quad n=1\,, \quad m=4\,, \quad l=\tfrac{1}{2} \quad \text{and} \quad [0,1]\in[0,(2E/m\omega^2)^{1/2}]\,, \tag{11}$$

which leads us to the following result:

$$L\geqslant 2\sum_{K=1}^{\infty}\frac{1}{2K}\,\frac{4^K\Gamma(4K-1)}{\Gamma(4K)}\,, \tag{12}$$

where Γ represents the factorial function

$$\Gamma(1+z)=z!\,, \tag{13}$$

such that

$$L\geqslant\sum_{K=1}^{\infty}\frac{4^K}{K}\,\frac{(4K-2)!}{(4K-1)!}=\sum_{K=1}^{\infty}\frac{4^K}{K}\,\frac{1}{4K-1}\,. \tag{14}$$

This numerical series diverges according to D'Alembert's convergence criterion [4]:

$$\lim_{K\to\infty}\frac{U_{K+1}}{U_K}=4\lim_{K\to\infty}\frac{K(4K-1)}{(K+1)(4K+3)}=4>1\,. \tag{15}$$

The result above indicates that the orbit length is a nonfinite quantity while the area (3) bounded by the same orbit remains finite. This indicates that the quantum field associated with the harmonic oscillator is suitable for a fractal description. Furthermore, since scale invariant properties of fractals are closely related to "chaos" and "strange attractors" in dynamical systems, it makes sense to attempt to fuse these concepts to SM.

3. The chaotic behavior of the global gauge transformation

Consider an isolated packet of waves representing an arbitrary free field $\phi(x)$. There are many examples of such "soliton-like" modes which can be configurated as local peaks. For instance one can take the ground state of the quantum harmonic oscillator [5]:

$$\phi^0(x) \sim \exp(-\tfrac{1}{2}x^2)\,, \tag{16}$$

where x stands for the field coordinate and the first order Hermite polynomial is

$$H_0(x) = 1\,. \tag{17}$$

Another traditional "soliton-like" wave can be introduced by performing a Fourier transformation on the space packet such that [6]

$$\phi_0(\boldsymbol{k}) \sim \int_D \mathrm{d}^3x \exp(-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x})\phi_0(x)\,. \tag{18}$$

With a translation of the wave number,

$$\boldsymbol{k}' = \boldsymbol{k} - \boldsymbol{k}_0\,, \tag{19}$$

where

$$k_0 = 2\pi/\lambda_0\,, \tag{20}$$

the function (18) has a unique maximum at $k' = 0$ and is symmetrical with respect to the origin:

$$\begin{aligned} &\partial\phi_0/\partial k' > 0 \qquad \text{for } k' < 0\,,\\ &\partial\phi_0/\partial k' < 0 \qquad \text{for } k' > 0\,. \end{aligned} \tag{21}$$

Consider now the standard gauge transformation which leaves the structure of the theory invariant [1] and apply this to (18):

$$\phi_0(\boldsymbol{k}) \to \phi_0'(\boldsymbol{k}) = \exp(-\mathrm{i}\chi)\,\phi_0(\boldsymbol{k})\,, \tag{22}$$

where χ is equivalent to an arbitrary "phase" factor. Notice that $\exp(-\mathrm{i}\chi)$ can be thought as a rotation operator in an appropriate space such that its components are always smaller or equal than unity. Since rotations generate a group of transformations with multiplication as a composition rule, a sequence of iterations applied to (22) will not alter its form. Consequently, from a physical standpoint, there is no way of separating "a priori" a first order iteration from a nth order one given by

$$\phi_0^{(n)}(k) = \underbrace{\phi_0 \cdot (\phi_0 \cdot (\phi_0 \cdot \cdots \cdot \phi_0(k)))}_{n \text{ times}} = \exp(-in\chi)\, \phi_0(k) . \tag{23}$$

On the other hand, transformation (22) defined above belongs to a larger class of mappings which display chaotic behavior. Under a given set of iterations, (22) may exhibit convergence to a stable attractor or erratic divergence depending on the phase selection. It is shown that a gradual variation of the phase factor drives a sequence of bifurcations such that, for each $j \geqslant 0$, function (22) has a single unstable orbit of period 2^j [2]. On letting the phase vary beyond a critical value the cycles generation replicates itself and (22) develops unstable orbits of period $3 \cdot 2^j$. The bifurcation scenario unfolds in a manner that makes it comparable to the concept of scale invariance derived from the geometry of fractal sets.

The chaotic behavior of the gauge transformation is similar to the divergence associated with the harmonic solution of the Klein–Gordon equation

$$(\partial^{\mu}\partial_{\mu} + m^2)\phi = 0 , \tag{24}$$

where the frequency is a complex number,

$$\omega = \omega_R + i\omega_I = \pm\sqrt{k^2 + m^2} \qquad (k = 2\pi/\lambda) , \tag{25}$$

such that $\omega_I = 0$ defines the marginal stability and $\omega_I > 0$ generates exponentially growing amplitudes [7].

The above considerations indicate that the non-univocity of phase choice leads to an internal instability of the gauge transformation. Therefore, the degeneracy of the gauge fields may be related to a "branching-out" pattern implied by the bifurcation mechanism. As a result, the operational equivalence of $\exp(-i\chi)$ and $\exp(-im\chi)$ may be regarded as the source of field architecture.

Furthermore, because gauging the elementary particles physics is the central idea of the Standard Model, we believe that a "chaotic" formalism can provide a simple and consistent description of the quantum field dynamics.

4. Postulates and conventions

To develop the frame of our approach the following statements are taken as assumptions:

$\langle P3\rangle$. Antisymmetric fields (labeling fermions) are generated or annihilated always in pairs. Consequently, the doublets are the fundamental eigenstates of the fermionic field, while singlets represent excited eigenstates.

$\langle P4\rangle$. Exclusion principle applies to all fermions.

$\langle P5\rangle$. *CPT* invariance is always valid when applied to either bosons or fermions while partial symmetries (like *CP*, *P*, *T*, . . .) are not necessarily valid and may be violated.

As a result of $\langle P5\rangle$, *CPT* invariance operates as a restriction rule which forbids some of the field states to appear as distinct (see section 5).

As a convention, the order in which the components of a doublet are listed corresponds to the time relationship between "cause" and "effect". Therefore, if T stands for the time inversion operator and (f_1, f_2) is a generic doublet, then

$$T(f_1, f_2) = (f_2, f_1)\,. \tag{26}$$

Finally, because the history of field transformation may be tracked in terms of phase selection (χ) or the number of cycles accumulated within the bifurcation process (N), either one of these two variables can be taken as an independent field coordinate.

5. The internal dynamics of the quantum field

The goal of this section is to generate and explain a map of field dynamics as outlined in section 3.

If one takes the period of cycles arisen from bifurcations as an input variable (N) and the phase χ as the output, then a plot of the field architecture looks like fig. 2.

In this map $V_1, V_2, V_3, \ldots$ or $V^1, V^2, \ldots$ stand for bifurcation vertices. Let us consider a generic field of form (18) being subjected to the bifurcation mechanism. For the time being, the particular structure of the field is not relevant so $\phi_0(\boldsymbol{k})$ may be scalar, spinorial, vectorial or tensorial. In each of vertices V_j or V^j $(j \leq N)$ transformation of the field must face the following "dilemma": if n_j is the number of cycles already generated and p_j is the number of new cycles, then the field behaves either symmetrically or antisymmetrically with respect to the transposition $n_j \leftrightarrow p_j$:

$$\phi(n_j, p_j) = \mathrm{e}^{\mathrm{i}\alpha}\phi(p_j, n_j)\,, \qquad \mathrm{e}^{\mathrm{i}\alpha} = \pm 1\,. \tag{27}$$

It seems natural to assign the symmetry of the field in (27) to bosons while

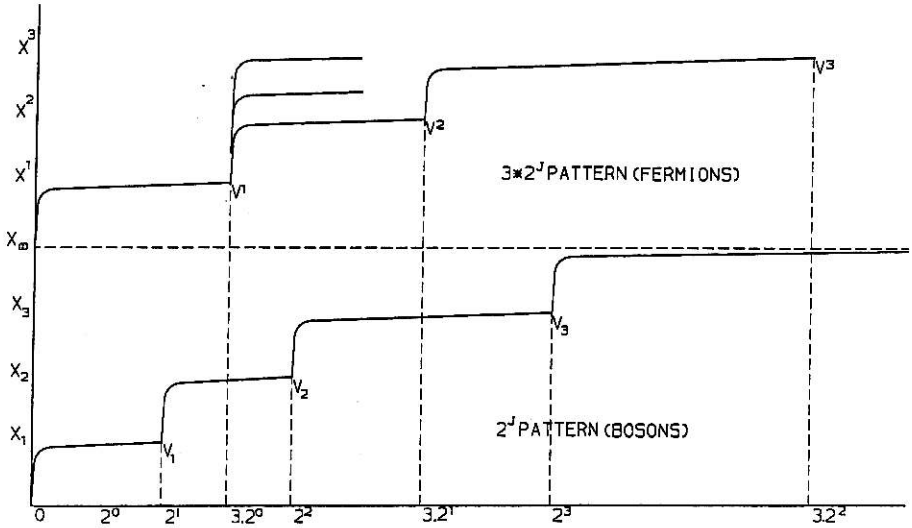

Fig. 2.

antisymmetry is associated with fermions. Since there is no privilege for either one of the two options, in each bifurcation vertex the field transformation may "jump" from the first branch of the plot to the second one. Therefore, if one considers the 2^j pattern of cycle accumulation typical for bosons and the $3 \cdot 2^j$ pattern typical for fermions, a transition boson–fermion appears to be a natural consequence of (27).

Let us discuss now each of the map vertices. Following the Standard Model, there are three symmetries to be taken into account for the gauge bosons: U(1), SU(2) and SU(3). Accordingly, three families of gauge bosons are introduced: the photon (γ), the weak triplet (W^+, W^-, W^0) and the multiplet of eight gluons ($g_1, g_2, \ldots, g_8$). The above map suggests a logical extension of this configuration as described below:

(a) In V_1 a number of 2^1 bosons are created and a natural partner for the photon (γ) could be the graviton (g).

(b) In V_2 a number of 2^2 bosons are created and the weak triplet may be replaced by two SU(2) weak triplets as follows:

$$V_2: \quad \zeta_1 = \begin{pmatrix} W^+ \\ W^- \\ W^0 \end{pmatrix}, \quad \zeta_2 = \begin{pmatrix} W^+ \\ W^- \\ B^0 \end{pmatrix}. \tag{28}$$

Here B^0 is the "extra" massive boson added to the initial family (W^+, W^-, W^0).

(c) In V_3 a number of 2^3 bosons are created and these may be identified with the gluon multiplet $(g_1, g_2, \ldots, g_8)$.

Following a similar judgement one can describe the structure of the fermionic vertices V^j.

(a) At V^1 the field triplicates because a number of $3 \cdot 2^0$ fermions are created. According to postulate ⟨P3⟩ these eigenstates must be doublets. Since a transition boson–fermion is likely to take place (see (27)) and V_1 is the nearest bosonic vertex, it makes sense to assume that V^1 is filled in with ultrarelativistic doublets (neutrino type),

$$V^1: \quad (\nu_e, \bar{\nu}_e), \quad (\nu_\mu, \bar{\nu}_\mu), \quad (\nu_\tau, \bar{\nu}_\tau). \tag{29}$$

We will focus below only on the electronic neutrino branch. Assuming the *CPT* invariance (postulate ⟨P5⟩), neutrino doublets appear only in a polarized state. If L and R stand for "left-hand" and "right-hand", then

$$CPT: \quad (\nu_e, \bar{\nu}_e)_L \rightarrow (\nu_e, \bar{\nu}_e)_R, \tag{30}$$

where the following line of operations was performed:

$$(\nu_e, \bar{\nu}_e)_L \xrightarrow{C} (\bar{\nu}_e, \nu_e)_L \xrightarrow{P} (\bar{\nu}_e, \nu_e)_R \xrightarrow{T} (\nu_e, \bar{\nu}_e)_R. \tag{31}$$

Consequently, the left-handed neutrinos and the right-handed ones are overlapped and only one polarized doublet has physical meaning. This conclusion agrees with the experimental data [1].

(b) At V^2 a number of $3 \cdot 2^1$ eigenstates develops. It makes sense to fill in this vertex with those leptons which make transitions from the neighbouring vertices possible. It appears that the electroweak interaction must play here a major role. A plausible configuration of V^2 may look like

$$\begin{aligned} V^2: \quad & (e^-, \bar{\nu}_e)_R, \quad (e^-, \ \ )_L, \quad (e^+, \nu_e)_L, \quad (e^+, \ \ )_R, \\ & (\ \ , e^-)_R, \quad (\ \ , e^+)_L. \end{aligned} \tag{32}$$

The first state is a SU(2) doublet while the second one is a SU(2) singlet. The reason for introducing such a singlet lies in the formal symmetry between $(e^-, \bar{\nu}_e)_R$ and $(e^-, \ \ )_L$. The component $\bar{\nu}_e$ is missing from the excited state $(e^-, \ \ )_L$ because left-handed antineutrinos are forbidden. The fourth and fifth states replicate the first and the second ones but are *not* identical with them. To check this statement one can apply a *CPT* operator such that

$$(e^-, \bar{\nu}_e)_R \xrightarrow{C} (e^+, \nu_e)_R \xrightarrow{P} (e^+, \nu_e)_L \xrightarrow{T} (\nu_e, e^+)_L, \tag{33}$$

$$(\nu_e, e^+)_L \neq (e^+, \nu_e)_L \, . \tag{34}$$

Also

$$(e^-, \quad)_L \xrightarrow{C} (e^+, \quad)_L \xrightarrow{P} (e^+, \quad)_R \xrightarrow{T} (\quad, e^+)_R \, , \tag{35}$$

$$(\quad, e^+)_R \neq (e^+, \quad)_R \, . \tag{36}$$

The fifth and sixth states enter as "mirror" images of electronic singlets as long as no privilege can be assigned to either left- or right-handed electrons. Relations (33) and (35) indicate that V^2 can be basically structured on three levels in an analogous manner as V^1, namely: V^2: (electron "up"; electron-neutrino "middle"; electron "down") because all the eigenstates listed in (32) are interchangeable with their symmetrical images. For instance,

$$(e^-, \bar{\nu}_e)_R \equiv (\nu_e, e^+)_L = T(e^+, \nu_e)_L \, , \tag{37}$$

$$(e^-, \quad)_L \equiv (\quad, e^+)_R = P(\quad, e^+)_L \, . \tag{38}$$

(c) At V^3 a number of $3 \cdot 2^2$ eigenstates develops which can be related to V^2 and V_3. Therefore, the structure of V^3 may be based on the quarks family and displays the following organisation:

$$V^3 : \begin{cases} (u_\alpha, d_\alpha)_L, & (u_\alpha, d_\alpha)_R, & (\bar{u}_\alpha, \bar{d}_\alpha)_L, & (\bar{u}_\alpha, \bar{d}_\alpha)_R \, , \\ (u_\beta, d_\beta)_L, & (u_\beta, d_\beta)_R, & (\bar{u}_\beta, \bar{d}_\beta)_L, & (\bar{u}_\beta, \bar{d}_\beta)_R \, , \\ (u_\delta, d_\delta)_L, & (u_\delta, d_\delta)_R, & (\bar{u}_\delta, \bar{d}_\delta)_L, & (\bar{u}_\delta, \bar{d}_\delta)_R \, . \end{cases} \tag{39}$$

Here (α, β, δ) is the triplet of colors carried by quarks such that

$$\alpha + \beta + \delta = 1 \, . \tag{40}$$

Let us point out that quarks enter *only* in doublets and cannot be therefore isolated as single excitations. Quark triplets appear as linear superpositions of these doublets.

Since every color has an anticolor ($\bar{\alpha}$, $\bar{\beta}$ or $\bar{\delta}$), *CPT* invariance must be operating as a prohibiting criterion for additional combinations in (39). In particular

$$(u_\alpha, d_\alpha)_L \xrightarrow{C} (u_{\bar{\alpha}}, d_{\bar{\alpha}})_L \xrightarrow{P} (u_{\bar{\alpha}}, d_{\bar{\alpha}})_R \xrightarrow{T} (u_{\bar{\alpha}}, d_{\bar{\alpha}})_R \, . \tag{41}$$

Notice here that C operates on the color only. If C would have been set to

Table I

Vertex	N	Configuration
V_1	2	$\gamma\, g$
V_2	4	$W^+\, W^0\, B^0\, W^-$
V_3	8	$g_1\, g_2, \ldots, g_7\, g_8$
V^1	3	$(\nu_\mu, \bar{\nu}_\mu)\,(\nu_e, \bar{\nu}_e)\,(\nu_\tau, \bar{\nu}_\tau)$
V^2	6	$(e^-, \bar{\nu}_e)_R\,(e^-,\ \ )_L\,(e^+, \nu_e)_L\,(e^+,\ \ )_R\,(\ \ , e^-)_R\,(\ \ , e^+)_L$
V^3	12	$(u_\alpha, d_\alpha)_L\,(u_\alpha, d_\alpha)_R \ldots (\bar{u}_\gamma, \bar{d}_\gamma)_L\,(\bar{u}_\gamma, \bar{d}_\gamma)_R$

operate on the quark itself, then

$$(u_\alpha, d_\alpha)_L \xrightarrow{CPT} (\bar{d}_\alpha, \bar{u}_\alpha)_R \neq (\bar{u}_\alpha, \bar{d}_\alpha)_R \,. \tag{42}$$

To conclude this section let us mention that the muonic and taonic branches separated at V^1 are equivalent to the electronic branch discussed above. Therefore, the field architecture stays invariant with respect to the substitution

$$(\nu_e, e, u, d) \rightarrow (\nu_\mu, \mu, c, s) \rightarrow (\nu_\tau, \tau, b, t) \,. \tag{43}$$

All the above results can be listed in one summary table as presented in table I.

6. A Lagrangian description of the bifurcation model

In the conventional formulation of the Standard Model, the full Lagrangian contains the operators of covariant differentiation in the kinetic term and mass + interaction contributions. For the bosons the mass term is $-m^2 b^2$ and the interaction term is linear in b. For the fermions both mass and interaction terms depend linearly on the mass and the interaction is also a linear expression of the four-vector current,

$$j^\mu = \bar{f}\gamma^\mu f \qquad (\mu = 0, 1, 2, 3) \,, \tag{44}$$

such that

$$\begin{aligned} L_b &= \tfrac{1}{2} D_\mu b D^\mu b - m^2 b^2 - b\rho(x^\mu) \,, \\ L_f &= \bar{f}\, i\gamma^\mu D_\mu f - \bar{f} m f + Q \bar{f}\gamma^\mu f A^\mu \,, \end{aligned} \tag{45}$$

where $L_b(L_f)$ stand for the bosonic (fermionic) lagrangian, $\rho(x^\mu)$ is the source of the scalar field while Q and A are, respectively, the charge and four-potential of the vector field.

To perform a complete description of a full Lagrangian, one would have to use a complicated formula taking into account all possible combinations between terms, such as

$$L_{\text{full}} = L_{\text{b}} + L_{\text{f}} + L_{\text{kinetic,bf}} + L_{\text{mass,bf}} + L_{\text{interaction,bf}} . \tag{46}$$

The full Lagrangian would have a composite structure including scalars, pseudoscalars, spinors or vectors.

For our model we will attempt to get the full Lagrangian starting from a simpler and more effective framework. To achieve this, several principles are to be considered:

(a) The approach is based upon a perturbative method.

(b) The full Lagrangian takes the phase χ as variable and the bosonic and fermionic fields as coefficients in the series expansion.

(c) The series contains powers less or equal to four. It can be shown that powers greater than four lead to infinities and are to be excluded [1].

(d) The interaction for both bosonic and fermionic fields is included in the quartic term of the series.

To derive the spectrum of the ground states arisen in a bifurcation vertex, one has to minimize the potential:

$$\partial L_{\text{int+mass}} / \partial \chi = 0 . \tag{47}$$

Following the standard procedure, further perturbations are defined by expanding the phase around the ground states. Accordingly, the ground states together with their perturbations fully specify the sequence of particles created as a result of bifurcation. An alternative expression for (23) is

$$\phi_0^{(n)}(k) = \sum_{j=0}^{\infty} \frac{(-i)^j (n\chi)^j}{j!} \phi_0(k) = \sum_{l=0}^{\infty} \phi_l(k) \xi^l , \tag{48}$$

where

$$j = 2l , \tag{49a}$$

$$\phi_l(k) = (-1)^l \phi_0(k) / (2l)! , \tag{49b}$$

$$\xi = n^2 \chi^2 . \tag{49c}$$

It is natural to assume that the general form of the function potential must be a power series of the field $\phi_0(\boldsymbol{k})$:

$$V = L_{\text{int+mass}} = \sum_{n=0}^{4} a_n \phi_0^n(\boldsymbol{k}) , \qquad a_n \in \mathbb{R} , \tag{50}$$

as long as the Lagrangian has to be invariant under the gauge transformation

$$V[\phi_0(\boldsymbol{k})] \to V[\exp(-\mathrm{i}\chi)\, \phi_0(\boldsymbol{k})] . \tag{51}$$

Replace now the terms in (50) by their gauge images given by (22),

$$\phi_0^n(\boldsymbol{k}) \to \exp(-\mathrm{i}n\chi)\, \phi_0^n(\boldsymbol{k}) = \exp(-\mathrm{i}n\chi)\, \phi_0(\boldsymbol{k})\, \phi_0^{n-1}(\boldsymbol{k}) , \tag{52}$$

or

$$\phi_0^n(\boldsymbol{k}) \to \phi_0^{(n)}(\boldsymbol{k})\, \phi_0^{n-1}(\boldsymbol{k}) , \tag{53}$$

where $\phi_0^{(n)}(\boldsymbol{k})$ represents the nth iterate of $\phi_0(\boldsymbol{k})$ (23).

(50) then becomes

$$V = \sum_{n=1}^{4} a_n \phi_0^{n-1}(\boldsymbol{k})\, \phi_0^{(n)}(\boldsymbol{k}) , \tag{54}$$

or

$$V(\xi) = \sum_{n=1}^{4} a_n \phi_0^{n-1}(\boldsymbol{k}) \left(\sum_{l=0}^{\infty} \phi_l(\boldsymbol{k}) \xi^l \right) . \tag{55}$$

Because the series must stop at the quartic term the exact formula (55) gives way to an approximate one:

$$V(\xi) \sim \sum_{l=0}^{4} a\phi_l(\boldsymbol{k}) \xi^l = \sum_{l=0}^{4} b_l \xi^l . \tag{56}$$

To get an acceptable level of confidence for the use of (56) one must normalize (49c) such that

$$|\xi| < 1 . \tag{57}$$

The interaction term is the last one ($l = 4$) and b_4 represents the strength coefficient for both fermions and bosons.

The mass term is a superposition of odd and even powers. Since the description is general, we would expect a symmetrical behaviour of the mass term for bosons and an antisymmetrical one for fermions. Therefore, (56) splits into two components,

$$\begin{aligned} V_{\mathrm{b}} &= b_{2,\mathrm{mass}}\xi^2 + b_{4,\mathrm{int}}\xi^4 , \\ V_{\mathrm{f}} &= b_0\xi^0 + b_{1,\mathrm{mass}}\xi + b_3\xi^3 + b_{4,\mathrm{int}}\xi^4 , \end{aligned} \tag{58}$$

$$\left.\begin{aligned} V_{\mathrm{b,mass}}(\xi) &= V_{\mathrm{b,mass}}(-\xi) \\ V_{\mathrm{f,mass}}(\xi) &= -V_{\mathrm{f,mass}}(-\xi) \end{aligned}\right. \quad \text{(at bifurcation vertices)} . \tag{59}$$

Under these circumstances (47) gives for bosons

$$\xi_{1,2} = \pm\sqrt{-b_2/2b_4} . \tag{60}$$

These are the vacuum expectation values of ξ and the treatment coincides with the spontaneously broken symmetry approach for the so-called Higgs field [1].

For fermions we get a cubic equation with the canonic form

$$\xi^3 + p\xi^2 + q\xi + r = 0 , \tag{61}$$

where

$$p = 3b_3/4b_4 , \qquad q = 0 , \qquad r = b_1/4b_4 \qquad (b_4 \neq 0) . \tag{62}$$

To check the nature of its solution, one has to evaluate the discriminant Q given by [8]

$$Q = \tfrac{1}{27}(-\tfrac{1}{27}p^6) + \tfrac{1}{4}(\tfrac{2}{27}p^3 + r)^2 , \tag{63}$$

or

$$Q = \tfrac{1}{27}p^3 r + \tfrac{1}{4}r^2 = r(\tfrac{1}{27}p^3 + \tfrac{1}{4}r) . \tag{64}$$

As long as $b_1 \neq 0$ $(r \neq 0)$ the cubic equation has three distinct roots which provide the vacuum expectation values for fermions.

The vacuum solutions correspond to the true stationary levels of the field subjected to the bifurcation mechanism. Therefore, it makes sense to think of them as of "massless" states associated with the field. Consequently, $\xi_{1,2}$ are connected with the photon and graviton (vertex V_1) while the three solutions of (61) relate to the ultrarelativistic neutrino doublets produced at vertex V^1.

Bifurcations evolve in a self-similar manner at next vertices repeating the pattern of V_1 or V^1. This is, for instance, the reason why at V_2 (and V^2) the map contains a duplicate (a triplicate, respectively) of the vacuum spectra,

$$V_2 : \begin{cases} (W^+, W^-) = |\uparrow\rangle , \\ (W^0, B^0) = |\downarrow\rangle , \end{cases} \qquad \text{(SU(2) doublet)} \tag{65}$$

$$V^2 : \begin{cases} [(e^-, \quad)_L , \quad (e^+, \quad)_R] = |\uparrow\rangle , \\ [(e^-, \bar{\nu}_e)_R , \quad (e^+, \nu_e)_L] = |0\rangle , \\ [(\quad , e^-)_R , \quad (\quad , e^+)_L] = |\downarrow\rangle . \end{cases} \qquad \text{(SU(2) triplet)} \tag{66}$$

As perturbations of the vacuum states, vertices V_j or V^j ($j \geqslant 2$) carry an inner excitation energy related to the shift from the ground level. Accordingly, each of the vertices V_j or V^j ($j \geqslant 2$) is the source of a unique mass spectrum. This circumstance supplies a plausible interpretation of how the mass should enter into the model.

7. Inertial frames under Lorenz invariance

The first segment of the bosonic branch (see fig. 2) represents the most familiar and simpler case of gauge invariance: the Lorenz transformation of inertial frames. It comes almost naturally to consider the covariance of the theory with respect to inertial frames as an immediate example of the gauge concept.

We will show in section 8 that an identical approach covers the gauge formalism of both electromagnetic and gravitational fields. Consequently, the Lorenz transformation is to be thought only as a first order approximation of the gauge formalism up to the first vertex (V_1).

Take two arbitrary inertial frames and let V stand for their relative linear velocity. In the Lorenz formula the rotational transposition of the coordinates operates as a 2×2 matrix,

$$\begin{pmatrix} t' \\ x' \end{pmatrix} = \begin{pmatrix} \cos \Psi & \sin \Psi \\ -\sin \Psi & \cos \Psi \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix} , \tag{67}$$

where Ψ is the imaginary rotation angle given by

$$\begin{gathered} V = \mathrm{i} \tan \Psi , \\ \sin \Psi = -\mathrm{i}V/\sqrt{1 - V^2} , \qquad \cos \Psi = 1/\sqrt{1 - V^2} . \end{gathered} \tag{68}$$

The space–time forms a continuous scalar field whose components are physically equivalent. Following a standard representation, define the complex scalar

$$\eta' = (t' + \mathrm{i}x')/\sqrt{2} = \mathrm{e}^{-\mathrm{i}\Psi}(t + \mathrm{i}x)/\sqrt{2} = \mathrm{e}^{-\mathrm{i}\Psi}\eta\,,$$
$$\eta'^{*} = (t' - \mathrm{i}x')/\sqrt{2} = \mathrm{e}^{\mathrm{i}\Psi}\eta^{*}\,. \tag{69}$$

Note that (69) is identical with (22). Consequently, the space–time can be treated as a regular pseudoscalar field. If m is the mass associated with it, the Lagrangian depends only on $\eta^*\eta$:

$$L = \partial_\mu \eta^* \partial^\mu \eta - m^2 \eta^* \eta \qquad (\mu = 0, 1)\,. \tag{70}$$

Two conserved "charges" emerge from Lorenz invariance of the field η:

$$\begin{aligned} &S_\mu = \mathrm{i}(\eta \partial_\mu \eta^* - \eta^* \partial_\mu \eta)\,, && \partial_\mu S_\mu = 0\,, && \\ &I = x^2 - t^2\,, && \partial_\mu I = 0 && \text{(interval)}\,. \end{aligned} \tag{71}$$

Let us recall now that the above relations (67)–(71) can be duplicated for the electromagnetic field under the following substitutions:

$$x^i \to A^i \qquad (i = 1, 2, 3)\,,$$
$$t \to \phi\,. \tag{72}$$

Therefore, there is no formal difference between the behavior of the space-time field (x^i, t) and the electromagnetic field (A^i, ϕ) with respect to the Lorenz transformation of inertial frames.

This circumstance gives us grounds to believe that the space–time and electromagnetic fields overlap in a gauge transformation theory. One may expect that these two fields separate at the ultrarelativistic limit, as long as $V \to c(=1)$ is the single instability associated with (68) and (69).

8. The ultrarelativistic scenario: photon–graviton split

As is well known, the Schrödinger equation describing the movement of a nonrelativistic spinless electron in an electromagnetic field,

$$[(1/2m_\mathrm{e})(-\mathrm{i}\nabla + eA)^2 - (\mathrm{i}\,\partial/\partial t + e\phi)]\,\Psi_\mathrm{e} = 0 \tag{73}$$

is left unchanged if one operates a gauge transformation on the electron wavefunction and on the field [1]:

$$\Psi'_e \rightarrow e^{-i\chi}\Psi_e\,, \tag{74a}$$

$$A'^{\mu} \rightarrow A^{\mu} - \partial^{\mu}\chi \qquad (e = 1)\,. \tag{74b}$$

We want to show that an identical picture can be developed for a particle of mass m submerged in a gravitational field. Following Einstein, a space–time wavefunction is to be attached to the particle:

$$\eta^{\mu} \leftrightarrow m\,, \tag{75}$$

which may be expressed in the standard exponential form

$$\eta^{\mu} = (\eta_0)^{\mu}\, e^{-i\rho_{\mu}}\,, \tag{76}$$

where $(\eta_0)^{\mu}$ is an amplitude and ρ_{μ} is the phase four-vector. Consequently, the equation of motion [9]

$$du_{\mu}/ds - \tfrac{1}{2}\,\partial_{\mu} g_{\nu\omega} u^{\nu} u^{\omega} = 0\,, \tag{77}$$

where u_{μ} is the four-velocity and $g_{\nu\omega}$ the metric tensor, would not be affected if one would replace the following items:

$$\begin{aligned} &\eta^{\mu} \rightarrow \eta'^{\mu}\,, \\ &g_{\nu\omega} \rightarrow g'_{\nu\omega}\,. \end{aligned} \tag{78}$$

The presence of the particle in the gravitational field is expected to produce a small perturbation of the latter. Therefore, a weak fluctuation of the space–time geometry is to be written as

$$\eta'^{\mu} = \eta^{\mu} + \delta^{\mu}_{\eta}\,, \tag{79a}$$

$$g'_{\nu\omega} = g_{\nu\omega} + \delta g_{\nu\omega}\,. \tag{79b}$$

Taking into account (76), (79a) can be reduced at a form similar to (74a),

$$\eta'^{\mu} \rightarrow (\eta_0)^{\mu}\, e^{-i(\rho_{\mu} + \delta\rho_{\mu})}\,. \tag{80}$$

(79b) leads to a first order variation of the metric tensor given by [9]

$$g'_{\nu\omega} \to g_{\nu\omega} - (\partial_\omega \tilde{\rho}_\nu + \partial_\nu \tilde{\rho}_\omega) , \qquad \tilde{\rho}_\mu = \exp(-\mathrm{i}\delta\rho_\mu) , \tag{81}$$

which is analogous with (74b).

Because we are operating in a non-Euclidian geometry, the weak fluctuations of the space–time have to be considered as covariant perturbations. Thus, the simple derivatives in (81) are to be replaced by the covariant ones whose iterations do not alter the gauge invariance of the theory.

Since (77) originates in cancelling out the covariant derivative of the four-velocity [9, 10],

$$\mathrm{D}u^\mu = 0 , \tag{82}$$

it is now obvious that a covariant perturbation of (82) induced by (80) would not change its form, i.e.

$$\begin{aligned} &\mathrm{D}(u^\mu + \delta u^\mu) = 0 , \\ &\mathrm{D}(\delta u^\mu) = 0 , \end{aligned} \tag{83}$$

or

$$\mathrm{D}(\delta u^\mu) = \mathrm{d}(\delta u^\mu) - \delta(\delta u^\mu) = 0 , \tag{84}$$

where δu^μ are the small fluctuations of the four-velocity.

Let us investigate now a further development of our approach to the Standard Model.

As long as the electromagnetic and gravitational fields seem to have a unique background originated in the gauge transformation, it makes sense to reveal a unique formula for the covariant derivative related to these fields.

For an electrically charged particle Q submerged in an electromagnetic field A^μ the covariant derivative is [1]

$$\mathrm{D}^\mu = \partial^\mu - \mathrm{i}QA^\mu . \tag{85}$$

Consider an arbitrary four-vector V^μ and recall its covariant derivative for a gravitational field:

$$\mathrm{D}V^\mu = (\partial_\omega V^\mu + \Gamma^\mu_{\nu\omega} V^\nu)\,\mathrm{d}x^\omega , \tag{86}$$

where $\Gamma^\mu_{\nu\omega}$ are the Christoffel symbols.

For the sake of simplicity let us set

$$|\mathrm{d}x^{\omega}| = 1\,, \tag{87}$$

and make the transposition [9]

$$V^{\nu} = V^{\mu}\delta^{\nu}_{\mu}\,, \tag{88}$$

where δ^{ν}_{μ} is the Kronecker symbol. (86) becomes

$$\mathrm{D}V^{\mu} = (\partial_{\omega} + \tfrac{1}{2}\delta^{\nu}_{\mu} g^{\mu\varepsilon}\Gamma_{\varepsilon,\nu\omega})V^{\mu}\,, \tag{89}$$

where

$$\Gamma_{\varepsilon,\nu\omega} = \partial_{\omega} g_{\varepsilon\nu} + \partial_{\nu} g_{\varepsilon\omega} - \partial_{\varepsilon} g_{\nu\omega}\,. \tag{90}$$

Here δ^{ν}_{μ} and $g^{\mu\varepsilon}$ are constants with respect to the covariant differentiation [9] and therefore they can be compared to the electrical charge in (85). Notice also that $\Gamma_{\varepsilon,\nu\omega}$ is written in the same form as the tensor of the electromagnetic field:

$$F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}\,. \tag{91}$$

The treatment outlined above suggests a physical interpretation of the weak isospin as well as a definition for the component W^0_{μ} of the SU(2) weak triplet $(W^{\pm}, W^0)$. To achieve this, let us analyze the second term of the covariant derivative expressed in the Standard Model as [1]

$$T_2 = -\tfrac{1}{2}\mathrm{i}g_1 Y B_{\mu}\,. \tag{92}$$

The massive bosonic field B_{μ} is a linear superposition of the electromagnetic field A_{μ} and of W^0_{μ},

$$B_{\mu} = \sqrt{1 + [(g_1/g_2)Y_{\mathrm{L}}]^2}\,A_{\mu} + (g_1/g_2)Y_{\mathrm{L}}W^0_{\mu} = c_{\mathrm{a}}A_{\mu} + c_{\mathrm{w}}W^0_{\mu}\,, \tag{93}$$

such that (92) becomes

$$T_2 = -\mathrm{i}g_1(Q - T_3)(c_{\mathrm{a}}A_{\mu} + c_{\mathrm{w}}W^0_{\mu})\,. \tag{94}$$

In an orthogonal representation, we can conveniently cancel out some components of the scalar product,

$$QW^0_{\mu} = T_3A_{\mu} = 0\,, \tag{95}$$

and (94) reduces to

$$T_2 = -\mathrm{i}g_1 c_{\mathrm{a}} Q A_{\mu} + \mathrm{i}g_1 c_{\mathrm{w}} Y T_3 W^0_{\mu} = T_{\mathrm{E}} + T_{\mathrm{G}} . \tag{96}$$

Since the first term is an electromagnetic contribution (according to (85)), the second term may be assigned to a gravitational one. Given the orthogonality condition, the gravitational component becomes

$$T_{\mathrm{G}} = -\mathrm{i}g_1 c^0_{\mathrm{w}} T^2_3 W^0_{\mu} \qquad (c^0_{\mathrm{w}} = 2g_1 g_2^{-1}) . \tag{97}$$

A comparison of (97) with the second term of (89) reveals the following similitudes:

$$T^2_3 \to \delta^{\nu}_{\mu} g^{\mu\varepsilon} , \quad W^0_{\mu} \to \Gamma_{\varepsilon,\nu\omega} . \tag{98}$$

This circumstance indicates that the isospin and the W particles are intimately connected with the gravitational field.

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